



## On The Performance Of Four Kinds of Chebyshev Polynomial in Numerical Treatment of Multi-Order Fractional Differential Equations

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### ABSTRACT

This paper is devoted to investigate the relationship in the performance of the four kinds of shifted Chebyshev polynomial for the comparison purpose, in fractional linear differential equation with constant co-efficient involving the Caputo fractional derivative, using Standard Collocation Method (SCM). Three different examples were considered and the results obtained show that the first kind of shifted Chebyshev polynomial performed better. Hence the order of performance is as follows: the first, fourth, second and third respectively.

**Keywords:** Multi-Order, Fractional Differential Equation, Chebyshev polynomials.

### INTRODUCTION

Multi-term fractional differential equation or composite fractional equations are known to be used in the modelling of visco-elasticity, electro-chemistry and material science (Milan, 1983, Vladimir, *et al.*, 1997 and Steven, 2007). Very well known examples of multi-term fractional differential equation include the basset equation (Milan, 1983, Vladimir, 1997 and Yuqiang, 2010) and the bagley-torvick equation (Richard, 1997). One very great constraint to the use of multi-term fractional differential equation in models is the existence of limited numerical schemes for their solution. Owing to the numerical applications of fractional differential equations in diverse fields, the solution techniques for fractional differential equation continue to attract renewed interest from researchers. A good number of researchers have proposed and applied some efficient approximation and analytical techniques for the solution of problems in fraction calculus. Such that techniques have been applied to fractional differential equation fractional integral equation, and in some cases fractional integro - differential equation. There have been attempts to solve multi-order fractional differential equations but a complete analysis has so far not given (Shafiqul, 2011, Kasi, *et al.*, 2010 and Das, ., 2010). Chebyshev polynomials are everywhere dense in numerical analysis. There is a scarcely any area of numerical analysis where chebyshev

polynomials do not drop in like surprise visitors, and indeed there are now a number of subjects in which these polynomials take a significant position in modern developments including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations. (Richard, 1997, Lima, 2000 and Al-Salam, 1986).

Based on the discrete orthogonality relationships of the Chebyshev polynomials, various methods for solving linear and nonlinear ordinary differential equations were devised at about the same time and were found to have considerable advantage over finite-difference methods ( Idress 2007, Rahman, *et al.*, 2012, and Stephen, 2004)

There are four (4) kinds of Chebyshev polynomials. But some books and many articles use the expression 'Chebyshev polynomial' to refer exclusively to the chebyshev polynomial  $T_n(x)$  of the first kind (Garcia-Amor, 2003). This work brings to light, the order of usefulness

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of the three kinds of Chebyshev polynomial and their performance.

The arrangement of this paper goes thus: In section 2 we review some basic definitions that will guide the readers in the understanding of the paper, in section 3 we demonstrate the numerical scheme on the general class of the problem, in section 4 we solve three numerical examples while in the 5th section we discuss the results and give the summary of the whole work.

**DEFINITION OF RELEVANT TERMS**

**Definition 1.** A real function  $f(x), x > 0$ , (Lewis,., 1991 and Arfken., 1985), is said to be in the space  $C_\mu, \mu \in \mathbf{R}$ , if there exists a real number  $P > \mu$  such that  $f(x) = x^P f_1(x)$ , where  $f_1(x) \in C[0,1]$

**Definition 2.** A function  $f(x), x > 0$ , (Shafiqul, 2011) is said to be in the space  $C_\mu^m, m \in \mathbf{N} \cup 0$ , if  $f^m \in C_\mu$

**Definition 3.** Caputo’s definition of the fractional - order derivative is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{1}$$

where  $n - 1 < \alpha \leq n, n \in \mathbf{N}$ ,  $\alpha$  is the order of the derivative and a is the initial value of function  $f$ .

For the Caputo’s derivative we have:

$$D^\alpha C = 0, C \text{ is a constant}$$

$$D_*^\alpha x^\beta = \begin{cases} 0 & , \beta < \alpha - 1 \\ \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha} & , \beta > \alpha - 1 \end{cases} \tag{2}$$

**Definition 4.** The Riemann - Liouville fractional integral operator of order  $\alpha \geq 0$ , of a function  $f \in C_\mu, \mu \geq -1$ , is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau \tag{3}$$

We mention only some properties of the operator  $J^\alpha$  : For  $f \in C_\mu, \mu, \gamma \geq -1, \alpha, \beta \geq 0$  :

$$J^0 f(t) = f(t), J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) = J^\beta J^\alpha f(t), \tag{4}$$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} t^{\gamma + \alpha} \tag{5}$$

Also, we need here two of its basic properties. If  $m - 1 < \alpha \leq m, m \in \mathbf{N}$ , and  $f \in C_\mu^m, \mu \geq -1$ , then

$$D^\alpha J^\alpha f(t) = f(t), J^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{t^i}{i!}, t > 0$$

Let M be the least common multiple of the denominators of  $\alpha, \beta_1, \dots, \beta_n$  and set  $\gamma = \frac{1}{M}$  and  $N = M\alpha$ .

**Definition 5: The first - kind polynomial  $T_n$**

The Chebyshev polynomial  $T_n(x)$  of the first kind is a polynomial in  $x$  of degree n, defined by the relation

$$T_n(x) = \cos n\theta \text{ when } x = \cos \theta \tag{6}$$

If the range of the variable  $x$  is the interval [-1, 1], then the range of the corresponding variable  $\theta$  can be taken as  $[0, \theta]$ . These ranges are traversed in the opposite directions, since  $x = -1$  corresponds to  $\theta = \pi$  and  $x = 1$  corresponds to  $\theta = 0$ .

**Definition 7: The second - kind polynomial  $U_n$**

The Chebyshev polynomial  $U_n(x)$  of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta} \text{ when } x = \cos \theta \tag{7}$$

**Definition 8:** The Chebyshev polynomials  $V_n(x)$  of the third order kinds is a polynomial of degree n in x defined respectively by

$$V_n(x) = \frac{\cos(n + \frac{1}{2})}{\cos \frac{1}{2} \theta} \tag{8}$$

**Definition 9:** The Chebyshev polynomials  $W_n(x)$  of the fourth order kinds is a polynomial of degree n in x defined respectively by

$$W_n(x) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2} \theta} \tag{9}$$

**SHIFTED CHEBYSHEV POLYNOMIAL**

Shifed Chebyshev Polynomial is  $x$  contained in (0, 1) which satisfies the recurrence formula:

$$T_{n+1}^* = (4x - 2)T_n^*(x) - T_{n-1}^*(x) \tag{10}$$

and is of four different kind which are stated in the next subsection

**3.1 Shifted Chebyshev Polynomial of the first kind**

Shifted Chebyshev Polynomial of the first kind denoted by  $T_n^*(x)$  are define as

$$T_n^*(x) = T_n^*(2x - 1) \tag{11}$$

**3.2 Shifted Chebyshev Polynomial of the second kind**

Shifted Chebyshev Polynomial of the second kind denoted by  $U_n^*(x)$  are define as

$$U_n^*(x) = U_n^*(2x - 1) \tag{12}$$

**3.3 Shifted Chebyshev Polynomial of the third kind**

Shifted Chebyshev Polynomial of the first kind denoted by  $V_n^*(x)$  are define as

$$V_n^*(x) = V_n^*(2x - 1) \tag{13}$$

**3.4 Shifted Chebyshev Polynomial of the fourth kind**

Shifted Chebyshev Polynomial of the first kind denoted by  $P_n^*(x)$  is defined as

$$W_n^*(x) = P_n^*(2x - 1) \tag{14}$$

over  $\text{CaCl}_2$  in a desiccator and weighed. [Weight = 22.0g and Yield = 69.2% ].

**STANDARD COLLOCATION METHOD**

In mathematics, a collocation method is a method for the numerical solution of ordinary differential equation and integral equations. The idea is to choose a finite dimensional space of candidate solution (usually polynomials up to certain degree) and a number of point in the domain (called collocation point) and to select that solution which satisfies the given equation at the collocation point. Now using the standard collocation method we consider the multi-fractional differential equation of the type

$$D^\alpha y(t) = \sum_{j=1}^k a_j(t) Dy(t) + a_0(t)y(t) + F(t) \tag{15}$$

with initial conditions

$$y^i(0) = b_i, i = 0, 1, \dots, [\alpha] - 1 \tag{16}$$

where  $\alpha > \beta_1 > \dots > \beta_k > 0$  are costant and  $\alpha$  denotes the smallest integer greater than or equal to  $\alpha$ . It should be noted that  $F$  can be linear in general, to solve equation (15) and (16) we approximate  $y(t)$ ,  $D^\alpha y(t)$  and  $t$  for  $i = 1, 2, 3, \dots, n$  using shifted Chebyshev polynomials. In order to use Chebyshev polynomials we assume our approximate solution  $y(t)$  as

$$y_n(t) = \sum_{i=0}^n a_i T_i \tag{17}$$

where  $n$  is the degree of the approximation used,  $T_i(x)$  is the four kinds of shifted Chebyshev polynomial basis and  $a_i$  is the coefficient of the series

Substituting (17) into (15) we have

$$D^\alpha \left( \sum_{i=0}^n a_i T_i(x) \right) = \sum_{i=0}^k a_i(t) D \left( \sum_{i=0}^n a_i T_i(x) \right) + a_0(t) \sum_{i=0}^n y(t) + F(t) \tag{18}$$

equation(18) after simplification becomes

$$D^\alpha (a_0 T_0(t) + a_1 T_1(x) + \dots + a_n T_n(x)) = \sum_{i=0}^k a_i(t) D(a_0 T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)) + a_0(t)(a_0 T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)) + F(t)$$

(19) equation (19) after a further simplification becomes

$$D^\alpha (a_0 T_0(t) + a_1 T_1(x) + \dots + a_n T_n(x)) - \sum_{i=0}^k a_i(t) D(a_0 T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)) - a_0(t)(a_0 T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)) + F(t) \tag{20}$$

collocating equation (20) after simplification, using

$$x_i = a + \frac{(b-a)i}{N} \quad i = 1, 2, 3, \dots, (n-1) \tag{21}$$

we have equation (20) after collocation represented in matrix form as

$$AX = Y \tag{22}$$

Where

$$A = \begin{pmatrix} D^\alpha T_0(x_1) - a_j(t) - D^{\beta_j} - a_0(t)T_0(x_1)D^\alpha T_0(x_1) - \sum_{j=0}^k a_j(t)D^{\beta_j} - a_0(t)T - 0(x_1) \\ D^\alpha T_0(x_2) - a_j(t) - D^{\beta_j} - a_0(t)T_0(x_2)D^\alpha T_0(x_2) - \sum_{j=0}^k a_j(t)D^{\beta_j} - a_0(t)T - 0(x_2) \\ \vdots \\ D^\alpha T_0(x_m) - a_j(t) - D^{\beta_j} - a_0(t)T_0(x_m)D^\alpha T_0(x_m) - \sum_{j=0}^k a_j(t)D^{\beta_j} - a_0(t)T - 0(x_m) \end{pmatrix} \tag{23}$$

$$X = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} \tag{24}$$

and

$$Y = \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_m) \end{pmatrix} \tag{25}$$

Equation (22) is solved simultaneously using maple software in order to obtain the unknown

constants which were then substituted into equation (17) to get the approximate solution.

**NUMERICAL EXAMPLES**

**Example 1**

Consider

$$D^2 y(t) + D^{\frac{3}{2}} y(t) + y(x) = 1 + x \tag{23}$$

together with initial conditions

$$y(0) = 0 \text{ and } y'(0) = 0 \tag{24}$$

The exact solution is

$$y(x) = 1 + x \tag{25}$$

**Solution**

The general form of shifted Chebyshev polynomial is

$$y(x) = \sum_{n=0}^M a_n x^n \tag{26}$$

solving for  $n = 0$  to  $M = 4$ , the approximate solution for first, second, third and fourth kind is respectively given as

$$y_1(x) = 1 + x$$

$$y_2(x) = 1 + x$$

$$y_3(x) = 1 + x$$

$$y_4(x) = 1 + x$$

**Example 2**

Consider

$$D^{\frac{1}{2}} y(t) + y(x) = x^2 + \frac{2x^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \tag{27}$$

together with initial conditions

$$y(0) = 0 \tag{28}$$

The exact solution is

$$y(x) = x^2 \tag{29}$$

**Solution**

The general form of shifted Chebyshev polynomial is

$$y(x) = \sum_{n=0}^M a_n x^n \tag{30}$$

solving for  $n = 0$  to  $M = 4$ , the approximate solution for first, second, third and fourth kind is respectively given as

$$y_1(x) = -9.68232366 \times 10^{-11} + 2.6 \times 10^{-9} x + 0.99999999846 x^2 + 3.098388077 \times 10^{-8} x^3 - 1.959173554 \times 10^{-8} x^4$$

$$y_2(x) = -7.65275034 \times 10^{-11} + 1.2 \times 10^{-9} x + 0.99999999913 x^2 + 1.979912998 \times 10^{-8} x^3 - 1.340656002 \times 10^{-8} x^4$$

$$y_3(x) = 1.374522065 \times 10^{-11} + 0.9999999924 x^2 + 1.866051332 \times 10^{-8} x^3 - 1.317189141 \times 10^{-8} x^4$$

$$y_4(x) = 7.97156556 \times 10^{-12} + 1.2 \times 10^{-9} x + 0.9999999904 x^2 + 2.191305016 \times 10^{-8} x^3 - 1.484072078 \times 10^{-8} x^4$$

**Example 3**

Consider

$$D^{\frac{3}{2}} y(t) = x^{\frac{3}{2}} y(x) + 4 \left( \frac{x^{\frac{1}{2}}}{\pi^{\frac{5}{2}}} \right) - x^{\frac{7}{2}} \tag{31}$$

together with initial conditions

$$y(0) = 0 \text{ and } y'(0) = 0 \tag{32}$$

The exact solution is

$$y(x) = x^2 \tag{33}$$

**Solution**

The general form of shifted Chebyshev polynomial is

$$y(x) = \sum_{n=0}^M a_n x^n \tag{34}$$

solving for  $n = 0$  to  $M = 4$ , the approximate solution for first, second, third and fourth kind is respectively given as

$$\begin{aligned}
 y_1(x) &= -1.0253503 \times 10^{-13} - 1 \times 10^{-10} x + 1.000000001 x^2 - 2.42362486 \times 10^{-9} x^3 + 1.38680348 \times 10^{-9} x^4 \\
 y_2(x) &= -6.420021054 \times 10^{-11} - 3 \times 10^{-10} x + 0.999999995 x^2 - 1.44048448 \times 10^{-9} x^3 - 1.19305547 \times 10^{-9} x^4 \\
 y_3(x) &= 1.42496628 \times 10^{-10} + 0.999999998 x^2 - 3.76836838 \times 10^{-10} x^3 + 4.68005489 \times 10^{-10} x^4 \\
 y_4(x) &= -5.19073123 \times 10^{-12} + 1.999999998 x^2 + 4.72407329 \times 10^{-9} x^3 - 3.79117286 \times 10^{-9} x^4
 \end{aligned}$$

**TABLE OF RESULTS**

Table 1: Table of Results for Example 1

x	Exact	First Kind	Second Kind	Third Kind	Fourth Kind
.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
.10	1.10000000	1.10000000	1.10000000	1.10000000	1.10000000
.20	1.20000000	1.20000000	1.20000000	1.20000000	1.20000000
.30	1.30000000	1.30000000	1.30000000	1.30000000	1.30000000
.40	1.40000000	1.40000000	1.40000000	1.40000000	1.40000000
.50	1.50000000	1.50000000	1.50000000	1.50000000	1.50000000
.60	1.60000000	1.60000000	1.60000000	1.60000000	1.60000000
.70	1.70000000	1.70000000	1.70000000	1.70000000	1.70000000
.80	1.80000000	1.80000000	1.80000000	1.80000000	1.80000000
.90	1.90000000	1.90000000	1.90000000	1.90000000	1.90000000
1.0	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000

Table 2: Table of Results for Example 2

x	Exact	First Kind	Second Kind	Third Kind	Fourth Kind
.00	0.00000000	$-9.68232366 \times 10^{-11}$	$7.628 \times 10^{-11}$	$1.3745 \times 10^{-10}$	$7.9716 \times 10^{-12}$
.10	0.01000000	0.01000000	0.01000001	0.01000001	0.01000000
.20	0.04000000	0.04000000	0.04000001	0.04000000	0.04000000
.30	0.09000000	0.09000000	0.09000001	0.08999998	0.09000000
.40	0.16000000	0.15999999	0.16000002	0.15999998	0.16000000
.50	0.25000000	0.25000001	0.25000002	0.24999997	0.25000000
.60	0.36000000	0.36000002	0.36000003	0.35999997	0.36000000
.70	0.49000000	0.49000001	0.49000002	0.48999996	0.49000000
.80	0.64000000	0.64000000	0.64000000	0.63999994	0.64000000
.90	0.81000000	0.80999934	0.80999998	0.80999989	0.80999996
1.0	1.00000000	0.99999984	0.99999986	0.99999978	0.99999982

Table 3: Table of Results for Example 3

x	Exact	First Kind	Second Kind	Third Kind	Fourth Kind
.00	0.00000000	$-1.0253503 \times 10^{-13}$	$-6.42002154 \times 10^{-11}$	$1.42496628 \times 10^{-10}$	$-5.19073123 \times 10^{-12}$
.10	0.01000000	0.00999998	0.01000001	0.01000001	0.01000000
.20	0.04000000	0.04000000	0.04000001	0.04000000	0.04000000
.30	0.09000000	0.09000000	0.09000001	0.08999998	0.09000000
.40	0.16000000	0.16000000	0.16000002	0.15999998	0.16000000
.50	0.25000000	0.24999999	0.24999998	0.25000001	0.25000000
.60	0.36000000	0.36000000	0.35999997	0.36000000	0.36000000
.70	0.49000000	0.48999999	0.48999997	0.49000000	0.49000000
.80	0.64000000	0.63999999	0.63999996	0.64000000	0.63999998
.90	0.81000000	0.80999998	0.80999996	0.80999999	0.80999997
1.0	1.00000000	1.00000000	1.00000000	1.00000000	0.99999992

## DISCUSSION OF RESULT

From the three tables above, it is clearly shown that the first and fourth kind of shifted Chebyshev polynomials performed better than the second and third kind of shifted Chebyshev polynomials

### Conclusion

So far, we have discussed how to solve multi-fractional differential equation using collocation method and the four kinds of shifted chebyshev polynomials and extracting out the approximate solutions of the problems considered. In conclusion, we observe that the first and the fourth kind shifted Chebyshev polynomials give a good approximate solution of the problem. The first and fourth kind of shifted chebyshev polynomials provide accurate, efficient, and good approximation of the multi-fractional differential equation.

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